Risk measures

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July 2010

Definition of risk and history of risk

- Risk: Hazard, a chance of bad cosequenes, loss or exposure to mischance,
- Risk: Chance taken,

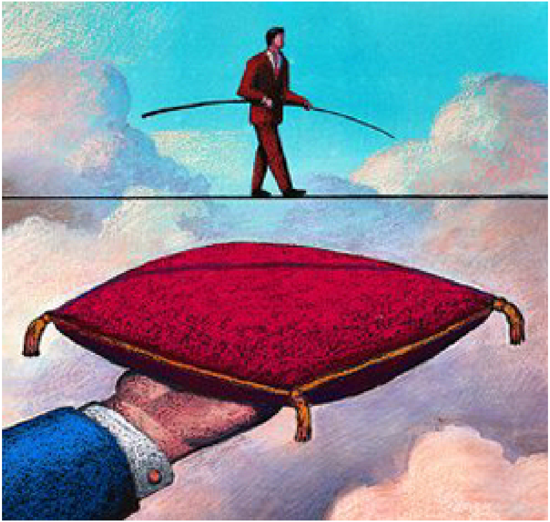
  Synonyms: accident, contingency, danger, exposedness, exposure, flyer, fortuity, fortune, gamble, hazard, header, jeopardy, liability, liableness, luck, openness, opportunity, peril, plunge, possibility, prospect, shot in the dark, speculation, stab, uncertainty, venture, wage

  Antonyms: certainty, safety, sureness, surety
- Certainty: belief

Concise Oxford English Dictionary, Roget's Thesaurus

Risk

Risk management



Short history of risk

- 2100 b.C. First description of a insurance contract for merchantise, Hamurabi code.
- Aristotle, first description of speculation in Politics, (real options)
- Genova 1347, maritime insurance, first use of the term "underwriter".
- Cardano 1501-1576: Liber de ludo alea (on games of chance)
- Galileo 1630: Sopre le scoperte dei dadi (on dice)
- Halley 1693: Pricing of life insurance, first mortality table .
- Founding of the first insurance company in 1688 (London, Lloyd's coffee house).
- 1787, 1794 New York insurance companies.

Parallel development of financial markets and first crises

- Defaults of England 1340, Spain etc
- Dutch tulip crisis 1637
- 1720 South sea crisis, 1720 Missisipi bubble, Panic of 1792 (New York), Panic of 1796-1797 (America-England)
- Danish state default 1813
- Panics of 1819, 1825, 1837, 1847, 1857, 1866, 1873, 1884, 1890, 1893, 1896 (mainly Englang, USA, Australia)
- 1901, 1907, 1910-11 (USA), 1910 (Sanghai), 1929 Wall Street crisis, 1973 Oil crisis, 1973-1974 (UK), 1980 (Latin America), 1983 (Israel), 1987 (Black monday), 1989-1991 (USA), 1990 (Japanese bubble), 1990 (Scandinavian bank crisis), 1992-1993 (Black Wednesday - speculative attacks on European currencies), 1994 (Mexico, speculative attacks on peso, default) 1997 (Asian crisis), 1998 (Russian crisis)
- 2001 (dot.com) , 2007-20?? The current crisis

Political, economic theorists, mathematicians and risk

- Pascal, Bernoulli, Condorcet, Laplace, Tetens, Lacroix
- Giddens, Beck, Mary Douglas ...
- Edgeworth, Wicksell, Fisher
- Knight, Keynes, Hicks, Chambers, Marschak, Savage
- Von Neumann, Morgernstern, ...
- Allais, Aumann, Gilboa, ...
- Markowitz, Roy, Tobin, Samuelson, Merton, ...
- Kolmogorov, Bachelier, Lévy, Itô, Borel, ...

Typology of risk

- Market risk
- Credit risk
- Liquidity risk
- Model risk

There is important feedback between the different types of risk
A holistic approach is necessary.

Description of risk

Must use probability theory.

- To describe uncertainty the following model is employed

  There exist S possible future statesof teh world; only one materializes

  Some of these states are more probable than others
- We must make the best decision under this uncertainty

Mathematical description

Sets of probable states of the world $\Omega = \{1, 2, \cdots, S\}$

$\sigma-$algebra on Ω

A map $P : \mathcal{F} \to [0, 1]$ describing how easy or not is for something in $\mathcal{F}$ to happen (probability measure).

P can be understood as scenarios concerning the future.

Proper choice of Ω and P is of paramount importance.

Assume risks can be parametrized by potential financial loss

A risk is understood as a map

$$X : \Omega \to \mathbb{R}$$

state of the world to loss (random variable).

If we face risks $X(t)$ at different times that we have a stochastic process

It is important to know $P(a \leq X(t) \leq)$.

Need for a probabilistic model either subjective or objective, to be confirmed by statistics.

Risk measures

For better understanding of risk must try to quantify risk by a single number

We are more used to comparing numbers than abstract entities such as random variables

Analogy with utility theory

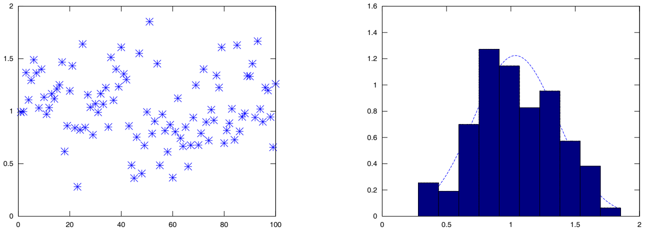
Dispersion

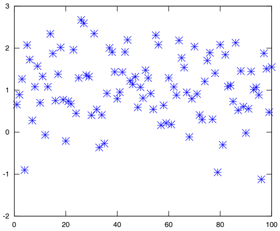
First measure that comes to mind:
Dispersion measures how far a single realization of the random variable may be from the expectation

$$Var(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$$

Dispersion was heavily used as a risk measure in the 1950s by Harry Markowitz (Nobel Laureate 1990) and became the cornerstone of Modern Portfolio Theory.

The larger the dispersion or returns, the riskier the investment

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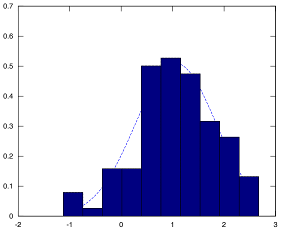
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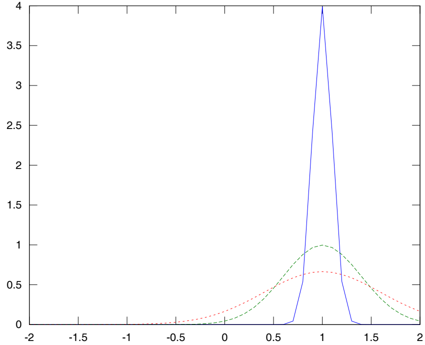
If returns follow $N(\mu, \sigma)$ then dispersion completely characterizes the return distribution.

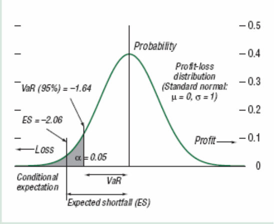


In reality returns almost never follow the normal distribution!

Value at risk VaR

Value at risk provides a measure which unlike dispersion may easily be translated to a pecuniary measure related to potential loss.

VaR is the largest loss that may appear for a given confidence level $a\%$, for a financial position of insurance portfolio



$$
\begin{aligned}
VaR_\alpha(L) &= \inf\{\ell \mid P(L > \ell) \leq 1 - \alpha\} \\
&= \inf\{\ell \mid P(L + \ell < 0) \leq \alpha\} \\
&= -\sup\{x \mid P(L < x) = \alpha)\}
\end{aligned}
$$

VaR may be interpreted as the minimum sum that must be added to a position so that it is accepteble with a given confidence level.
Capital adequacy measure for banks, insurance companies etc.

First used after the 1987 crisis although the idea preexisted.

VaR became the basic measure to quantify market risk within Basel II.

It is extended to insurance companies(Solvency II)

Computation of VaR

VaR may be computed with any statistical method that allows to estimate quantiles.

- Historical approach: Find the empirical distribution and through that calculate the quantiles.
- Parametric methods, e.g. assume a class of distributions and use estimators for the parameters.
- Non parametric methods, e.g. kernel estimators

Parametric models

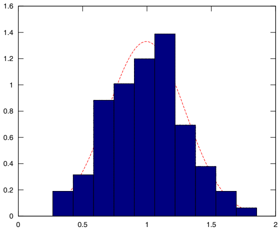
- Suppose we have historic data from the log returns of an investment $X(i)$, $i = 1, \cdots, N$
- Assume a model for the returns e.g. $X \sim N(\mu, \sigma)$
- Estimate the model parameters

$$\mu \simeq \hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X(i)$$

$$\sigma^2 \simeq \hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X(i) - \hat{\mu})^2$$

- From the parametric form of the distribution estimate the distribution and the quantiles, VaR

Very sensitive method with respect to model risk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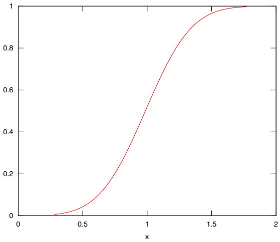
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Non parametric models: Kernel estimator

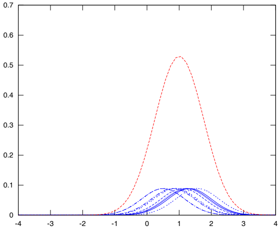
- Suppose we have historic data $X(i)$, $i = 1, \cdots, N$
- From these data we reconstruct the distribution $\hat{f}(x)$ of returns
- Select a kernel function $K(x)$, and a smoothing parameter h
- For each of the data $X(i)$ compute

$$K_i(x) = K\left(\frac{x - X(i)}{h}\right)$$

- Estimate $\hat{f}(x)$ as

$$\hat{f}(x) = \frac{1}{h}\sum_{i=1}^{N} K_i(x)$$

- From that estimate VaR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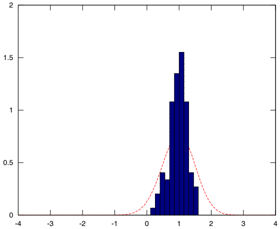
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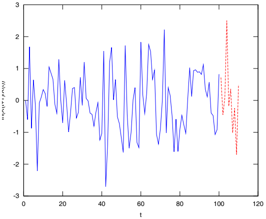
Pitfalls of the historic approach

If we calculate VaR for a position whose returns are given in the following time series

- Using the historic approach on the data for the days 1-100 we calculate VaR on day 101 to be 1.645/unit.
- In reality Var is 3.28!
- This is because from day 100 onwards the law of the process has changed but the historic approach has not yet adjusted to that!
- This approach can be very unsatisfactory for data exhihbiting change points.



Critique of VaR

- VaR is not always an adequate risk measure when returns deviate from normal distribution.
- The problems become attenuated when strong correlations are present
- Provides illusions of certainty, which may be catastrophic especially when combined with model uncertainty
- Counter intuitive results – It may disencourage diversification
- Good tool, but may become fatal in the wrong hands

Tail VaR

Tail Var quantifies expected loss given that yhe loss that occurs exceeds VaR

$$TVaR_\alpha(L) = \mathbb{E}[L \,|\, L \geq VaR_\alpha(L)]$$

For continuous distribution

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\beta(L)\, d\beta = TVaR_\alpha$$

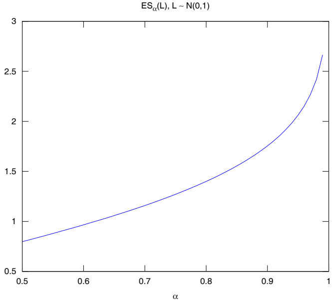
Expected shortfall: Average VaR over different confidence levels from α upwards

This resolves the problem of confidence level choice – Robustness

TVaR (ES) for various parametric models

$L \sim N(\mu, \sigma)$

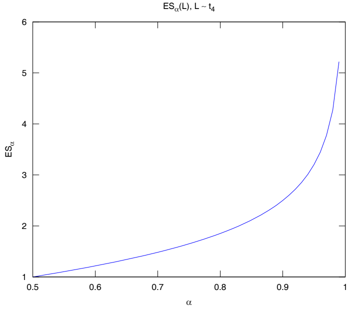
$$ES_\alpha = \mu + \sigma \frac{f(F^{-1}(\alpha))}{1-\alpha}$$
$$f : pdf, \quad F : cdf$$



If $\bar{L} = \frac{L-\mu}{\sigma} \sim T_\nu$ follows Student with ν degrees of freedom)

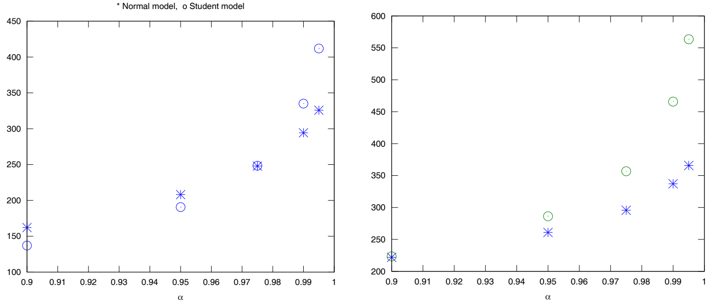
$$ES_\alpha(\bar{L}) = \frac{t(T^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (T^{-1}(\alpha))^2}{\nu - 1} \right)$$

Student distribution is heavy tailed



Calculation of Var_α and ES_α for the same position under different models

Suppose we calculate Var_α and ES_α for the same position under different models: the normal distribution and the Student distribution



This is a good example of model risk

Coherent risk measures

$\mathcal{L}$ set of risks
We consider a risk measure as a mapping $\rho : \mathcal{L} \to \mathbb{R}$.
This mapping should axiomatically have certain properties

- For every $L \in \mathcal{L}$ and every $\ell \in \mathbb{R}$ it holds $\rho(L + \ell) = \rho(L) + \ell$.
- For every $L_1, L_2 \in \mathcal{L}$ it holds $\rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2)$.
- For every $L \in \mathcal{L}$ and $\ell \in \mathbb{R}^+$ it holds that $\rho(\lambda\, L) = \lambda\, \rho(L)$
- For every $L_1, L_2 \in \mathcal{L}$ such that $L_1 \leq L_2$ it holds that $\rho(L_1) \leq \rho(L_1)$.

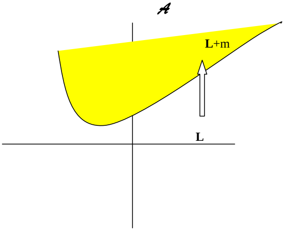
A mapping ρ satisfying these axioms is called coherent risk measure.

Economic interpretation of the axioms

- Axiom 1 guarantees existence of ceratainty equivalen – capital adequacy – as $\rho(L)$, since

$$\rho(L + \rho(L)) = 0$$

- There exists a convex subset $\mathcal{A} \subset \mathcal{L}$ containing all acceptable risks (acceptance set) defined as $\mathcal{A} = \{L \in \mathcal{L} \,|\, \rho(L) \leq 0\}$.
- If $L \notin \mathcal{A}$ then $L + m = L + \rho(L) \in \mathcal{A}$.



- Second axiom – subadditivity – is related to diversification (Markowitz).
- If a non subadditive measure is used then it may provide incentives to break an institution to several subsidiaries so as to reduce adequacy level
- Axioms 2 and 4 are related since for every $n \in \mathbb{N}$

$$\rho(n\,L) = \rho(L + \cdots + L) \leq n\,\rho(L)$$

- Axiom 4 (monotonicity) tells us that riskier positions should have larger risk measures and thus larger adequacy levelsT

Convex risk measures

An interesting variation of coherent risk measures

- These came out of the critique of Axiom 3: If $\rho(\lambda L) \leq \lambda \rho(L)$ then an institution prefers large portfolios and this may lead to liquidity problems!

Replace this axiom by convexity

$$\rho(\lambda L_1 + (1-\lambda) L_2) \leq \lambda \rho(L_1) + (1-\lambda) \rho(L_2), \;\; \lambda \in [0,1]$$

- Convex and coherent risk measures are very recent – they were proposed in 1999-2000
- However, they stirred great interest
- There is already an ambitious plan in progress to restate portfolio theory and economic theory in general (e.g. general equilibrium theory) substituting the concept of utility function by the concept of risk measure
- Applications in insurance

Examples of coherent or convex risk measures

Example

Suppose that L is a risk and let us take the map ρ_{max} defined by

$$\rho_{max}(L) = sup_{\omega \in \Omega} |L(\omega)|$$

This is the worst possible loss

ρ_{max} is a coherent risk measure and it is the most conservative conceivable risk measure since for any other risk measure it holds that

$$\rho(L) \leq \rho_{max}(L)$$

Example

Tail VaR or equivalently Expected shorfall are coherent risk measures
VaR is not!

Example

Consider an expected utility function u.
For some risk L let us calculate $\mathbb{E}[u(L)]$
The certainty equivalent

$$\rho(L) := u^{-1}(\mathbb{E}[u(L)])$$

is a convex risk measure.

Why VaR is not a coherent risk measure

Suppose we have corporate bonds of 100 firms $i = 1, \cdots 100$ each with 2% probability of default.

Each bond costs 100 and pays 105 in case of no default and 0 otherwise.

The loss L_i from each bond is

$$L_i = \begin{cases} -5 & \text{no default} \quad P(L_i = -5) = 0.98 \\ 100 & \text{default} \quad P(L_i = 100) = 0.02 \end{cases}$$

(negative loss means profit)

Assume that the default events are independent

Let us compare two portfolios

- A: 100 bonds of firm 1
- B: Completely diversified portfolio, 1 bond from each firm (total 100 bonds)

Our experience shows that the completely diversified portfolio is safer – therefore must have smaller VaR

- However, $L_A = 100\, L_1 \implies VaR_{0.95}(L_A) = 100\, Var_{0.95}(L_1)$
Since $VaR_{0.95}(L_1) = -5$, we see $VaR_{0.95}(L_A) = -500$, so position A does not require any capital adequacy at all!

- For portfolio B

$$L_B = \sum_{i=1}^{100} L_i = 105 \sum_{i=1}^{100} Y_i - 500$$

where $P(Y_i = 1) = 0.98, P(Y_i = 0) = 0.02$ (indicator function of default).
$M := \sum_{i=1}^{100} Y_i \sim B(100, 0.02)$
A simple calculation shows that $VaR_{0.95}(L_B) = 25$
This position is assumed risky and we need to input 25 monetary units to undertake it.

Therefore, using VaR the completely diversified position B is considered as riskier than the fully concentrated position A!
This is counter intuitive, and contrary to diversification as well as subadditivity since if we had a subadditive risk measure

$$\rho(L_B) = \rho(\sum_{i=1}^{100} L_i) \leq \sum_{i=1}^{100} \rho(L_i) = 100\,\rho(L_1) = \rho(L_A)$$

Therefore VaR is not subadditive and is not a coherent risk measure!

This disturbing propery of VaR may come up if there are heavy tails in the rsk distribution, something very common in practice!

VaR is subadditive if risks follow elliptic distributions.

However, this is not always the case!

Subjective probability

- The scenarios Q concerning future states of the world may be objective or subjective
- Risk and Uncertainty
- Weather phenomena, roullete or certain financial phenomena may be described by objective probabilities
- A political crisis, an agreement between two firms, the probability of an employer making a trivial mistake etc. may be described by subjective probabilities.
- it is possible to extend the framework of probabiity theory so as to include the concept of subjective probability and how this may interact with real measurements etc.

Subjective probability and economic theory

The concept of subjective probability has been used extensively in economic theory.
Fundamental contributions from Knight, Savage, Auman, Schmeidler, Gilboa et al

Coherent risk measures and the return of subjective

- In a complete market in equilbrium the concept of subjective no longer is meaningfull as all agents are compelled to adopt the same (risk neutral) measure concerning future state of the world.
- However, markets are seldom complete! Non completeness allows the existence of different scenarios and even in equilibrium agents are allowed to adopt different beliefs.
- How is this subjectivity reconciled with the measurement of risk?

Let $\mathcal{Q}$ be the set of all possible scenarios concerning future states of the world.

Theorem

The map $\rho_L = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[L]$ i.e., the worst expected loss over all possible scenarios is a coherent risk measure.
Conversely, any coherent risk measure ρ can be represented as $\rho_L = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[L]$ for some set of possible scenarios $\mathcal{Q}$.

This theorem (Follmer, Schied, Artzner) is very important since it marks the return of the subjective in the description and measurement of risk and reconciles it with coherence.

For convex risk measures this becomes even more interesting

Theorem

Every convex risk measure ρ can be expressed as

$$\rho(L) = \sup_{Q \in \mathcal{Q}} \left(\mathbb{E}_Q[L] - a(Q) \right)$$

where $a : \mathcal{Q} \to \mathbb{R}$ is a penalty function.

For instance this penalty function gives very little weight to events that are highly improbable.

Risk measures, model uncertainty and robust preferences

Assume that an investor must choose bewteen 3 choices

A: Returns 1000 with probability 50% and -1000 with probability 50%

B: Returns 0 (the mean A) with certainty.

C: Returns 1000 with probability p and -1000 with probability $1 - p$ but p is unknown.

This investor must face uncertainty with respect to the probabilistic model that describes returns.
Knightian uncertainty
If the agent is ambiguity averse she will try to restrict choice between A and B
If agent is risk averse will prefer B.

Risk measures allow us to include in decision theory cases where we have uncertainty of this type

Robust preferences

$$U(L) = \inf_{Q \in \mathcal{Q}} \mathbb{E}_Q[u(L)]$$

These preferences are not based on a single model but take into account large families of models, therefore, they tackle model uncertainty very effectively.

Robust preferences were proposed by Gilboa and Schmeidler to resolve paradoxes in choice theory such as the Elsberg, Allais, or Kahneman paradoxes etc.

This representation is related to risk measures and introduces the concept of robustness of risk measurement with respect to model uncertainty.

We may use such measures in order to write down and solve decision problems under the effect of model uncertainty, using convex analysis and duality techniques.

$$\inf_{A} \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[L^A]$$

A a set of actions

The statement and resolution of such problems will give us more realistic descriptions concerning decision making when there is considerable uncertainty about the model we need to describe risk

Such techniques have been applied to the pricing of derivatives in cases where the model for the underlying is uncertain.

This is a very important problem for instance in march 1997 Mitsubishi bank announced losses of 83.000.000 $ due to bad choice of pricing model for swaps, while in the same year NatWest anounced lossed of 50.000.000 £for similar reasons.

Convex and coherent risk measures may allow us to quantify such risks and thus make a better job in understanding them as well as managing them.

Connections with ERM?

Can convex and coherent risk measures be integrated into ERM?
They can be very useful in the description and measurement of the various risks.
Because of the holistic nature of ERM certain risks are difficult to be quantified and subjective approaches must be taken in their description.
At the same time other risks can be well documented and subject to well defined statistical tools.
The representation of coherent risk measures in terms of scenarios and their robustness features allows us to use them to treat in an integrated fashion the diverse features present in ERM model.

Coherent risk measures provide a well founded axiomatic framework that allows the description and measurement of a wide spectrum of risks

- They bring back the subjective element in the measurement and description of risk
- They can handle problems of model uncertainty as well as lack of data and changes in law.
- They can handle extreme distributions
- They ecourage diversification even in extreme conditions

They will definitely play an important rôle in the risk management in the 21st century.

FURTHER READING

H. Fölmer and A. Schied, Mathematical Finance, De Gruyter, 2000
A. McNeil, R. Frey and P. Embrechts, Quantitative Risk Management, Princeton, 2005

Thank you for your attention!